The Axiom of Choice and the Banach-Tarski Paradox

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Axiom of Choice (Suppes 1960)

For any set A there is a function f such that for any non-empty subset B of A, $f(B) \in B$.

Choosing a shoe from each of an infinite set of pairs of shoes does not require the axiom of choice: just choose the left shoe.

Choosing a sock from each of an infinite set of pairs of socks requires the axiom of choice.

What makes AC so special?

- AC is *consistent* with ZF (Gödel 1935). Thus, if the axioms of ZF do not result in any contradictions, ZFC is also free of contradictions.
- AC is *independent* of ZF (Gödel, Cohen 1963). Thus, AC cannot be proved true or false using the axioms of ZF.
- AC is *non-constructive*. It guarantees the existence of a choice function but provides no method for constructing such a function.
- AC can be used to prove counter-intuitive results, like the Banach-Tarski paradox.

Rigid Motions and Congruence in $\mathbb{R}^3$

Definition

A **rigid motion** is a mapping from $\mathbb{R}^3$ to $\mathbb{R}^3$ that is a translation, rotation, or combination of the two (but not a reflection).
If a set Y is an image of a set X under some rigid motion, then we say X is **congruent** to Y.

Definition

A **partition** of a set X is a collection of pairwise disjoint sets $\{X_i\}_{i=1}^n$, for some $n \in \mathbb{N}$, such that $X = \bigcup_{i=1}^n \{X_i\}$.

Equidecomposability

Definition

Two sets X, Y are **equidecomposable**, denoted $X \approx Y$, if and only if there exist partitions $\{X_i\}_{i=1}^n$, $\{Y_i\}_{i=1}^n$ such that $X = \bigcup_{i=1}^n \{X_i\}$, $Y = \bigcup_{i=1}^n \{Y_i\}$, and X_i is congruent to Y_i for each $i = 1, \ldots, n$.

The above two figures are equidecomposable.

Equidecomposability

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Lemma

The following hold:

1. $\approx$ *is an equivalence relation.*
2. *If X and Y are disjoint unions of X_1, X_2 and Y_1, Y_2, respectively, and if $X_i \approx Y_i$ for each $i = 1, 2$, then $X \approx Y$.*
3. *If $X_1 \subseteq Y \subseteq X$ with $X \approx X_1$, then $X \approx Y$.*

The Banach-Tarski Paradox

Theorem (The Banach-Tarski Paradox, 1924)

Let U be a closed ball. Then there exists a partition $\{X, Y\}$ of U such that $U \approx X$ and $U \approx Y$.

- A "strong form" of the paradox that applies to *any* closed region with non-empty interior.
- The decomposition uses only *finitely many* pieces.
- Holds in more than three dimensions.
- Holds in one and two dimensions only if *countably many* pieces are used.

The Secret Ingredient: Hausdorff's Paradox

Theorem (Hausdorff's Paradox, 1914)

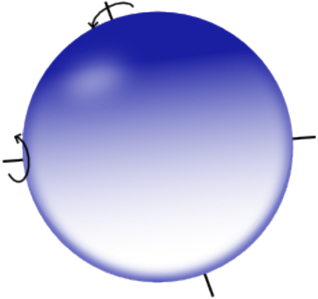
A sphere S can be decomposed into disjoint sets

$$S = A \cup B \cup C \cup Q$$

such that:

1. *the sets A, B, C are congruent to each other;*
2. *the set $B \cup C$ is congruent to each of the sets A, B, C;*
3. *Q is countable.*

Take Two Rotations and Call Hausdorff in the Morning



- Let φ be a rotation of 180° about some axis a_φ.
- Let ψ be a rotation of 120° about some axis a_ψ.
- Then $\varphi^2 = \psi^3 = 1$.
- Let G be the free product of the groups $\{1, \varphi\}$ and $\{1, \psi, \psi^2\}$.
- Elements of G (*words*) look like this:

$$\varphi\psi^{\pm 1} \dots \varphi\psi^{\pm 1}$$

(since $\psi^2 = \psi^{-1}$).

A Partition of G

- We construct a partition $\{G_1, G_2, G_3\}$ of G such that

$$G_1 \cdot \varphi = G_2 \cup G_3, \qquad G_1 \cdot \psi = G_2, \qquad G_1 \cdot \psi^2 = G_3.$$

- Let $1 \in G_1, \qquad \varphi, \psi \in G_2, \qquad \psi^2 \in G_3$.
- Let $\alpha \in G$. We assign $\alpha\varphi$ and $\alpha\psi$ to partitions based on cases.
  1. α ends with ψ or ψ^2
     - If $\alpha \in G_1$, put $\alpha\varphi$ into G_2.
     - Otherwise, put $\alpha\varphi$ into G_1.
  2. α ends with φ
     If $\alpha \in G_i$, put $\alpha\psi$ into $G_{i+1 \pmod 3}$ and $\alpha\psi^2 \in G_{i+2 \pmod 3}$.

Fixed Points and Orbits

Definition

For a rotation $\alpha \in G$, a point $x \in S$ is **fixed** if $x \cdot \alpha = x$.

- Let Q be the set of all fixed points under all rotations of G.
- Each rotation has two fixed points (one at either “pole”).
- Thus, Q is countable.

Definition

For each $x \in S \setminus Q$, the **orbit of** x, P_x, is the image of x under every rotation in G:

$$P_x = \{x \cdot \alpha \mid \alpha \in G\}.$$

Invoking the Axiom of Choice

- We can show that the orbits P_x for all $x \in S \setminus Q$ form a partition of $S \setminus Q$.
- Let M be the set containing exactly one point from each orbit.
- Each orbit contains uncountably many points. The collection of these orbits is more like a collection of pairs of socks than pairs of shoes.
- So, to guarantee that M exists, we need the axiom of choice.

The Secret Ingredient: Hausdorff's Paradox

Theorem (Hausdorff's Paradox, 1914)

A sphere S can be decomposed into disjoint sets

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such that:

1. *the sets A, B, C are congruent to each other;*
2. *the set $B \cup C$ is congruent to each of the sets A, B, C;*
3. *Q is countable.*

- We already have Q, the set of all fixed points of rotations in G. Now we just need A, B, and C.

A Decomposition of the Sphere

- Let $A = M \cdot G_1$, $B = M \cdot G_2$, $C = M \cdot G_3$.
- We've used M to partition $S \setminus Q$ according to our partition of G.
- Recall

$$G_1 \cdot \varphi = G_2 \cup G_3, \qquad G_1 \cdot \psi = G_2, \qquad G_1 \cdot \psi^2 = G_3,$$

so

$$A \cdot \varphi = B \cup C, \qquad A \cdot \psi = B, \qquad A \cdot \psi^2 = C.$$

- Hence, A, B, C, and $B \cup C$ are congruent to each other, and

$$S = A \cup B \cup C \cup Q.$$

Decomposing the Closed Ball

Theorem (The Banach-Tarski Paradox)

Let U be a closed ball. Then there exists a partition $\{X, Y\}$ of U such that $U \approx X$ and $U \approx Y$.

- Let $X \subset S$. Let $\mathcal{X}$ denote the set of all $x \in U$, other than the centre, such that the projection of x onto the surface is in X.
- Then from $S = A \cup B \cup C \cup Q$ we obtain

$$U = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{Q} \cup \{c\},$$

where c is the centre.

Decomposing the Closed Ball

$$S = A \cup B \cup C \cup Q, \quad U = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{Q} \cup \{c\}$$

- From the congruence of A, B, C and $B \cup C$, we have

$$\mathcal{A} \approx \mathcal{B} \approx \mathcal{C} \approx \mathcal{B} \cup \mathcal{C}.$$

- Let $X = \mathcal{A} \cup \mathcal{Q} \cup \{c\}$ and $Y = U \setminus X$.

Lemma(2)

If X and Y are disjoint unions of X_1, X_2 and Y_1, Y_2, respectively, and if $X_i \approx Y_i$ for each $i = 1, 2$, then $X \approx Y$.

- So $\mathcal{A} \approx \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$, and thus

$$X \approx U.$$

Showing $Y \approx U$

$$S = A \cup B \cup C \cup Q, \quad U = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{Q} \cup \{c\}$$

$$X = \mathcal{A} \cup \mathcal{Q} \cup \{c\}, \quad Y = U \setminus X$$

- We have partitioned the ball U into X and Y and shown $U \approx X$. Now we have to show $U \approx Y$.
- Start by taking a rotation $\tau \notin G$ such that $Q \cap (Q \cdot \tau) = \emptyset$
- No fixed point of any rotation in G is fixed under τ, so

$$Q \cdot \tau \subseteq A \cup B \cup C.$$

Showing $Y \approx U$

$$S = A \cup B \cup C \cup Q, \quad U = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{Q} \cup \{c\}$$

$$X = \mathcal{A} \cup \mathcal{Q} \cup \{c\}, \quad Y = U \setminus X$$

- No fixed point of any rotation in G is fixed under τ, so

$$Q \cdot \tau \subseteq A \cup B \cup C,$$

- Also, $\mathcal{C} \approx \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$.
- Then there exists $T \subseteq C$ such that T is congruent to $Q \cdot \tau$ and thus to Q. Hence, $\mathcal{T} \approx \mathcal{Q}$.

Showing $Y \approx U$

$$S = A \cup B \cup C \cup Q, \quad U = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{Q} \cup \{c\}$$

$$X = \mathcal{A} \cup \mathcal{Q} \cup \{c\}, \quad Y = U \setminus X$$

- Take any $p \in \mathcal{C} \setminus \mathcal{T}$. Then:

$$\mathcal{A} \approx \mathcal{B}, \quad \mathcal{Q} \approx \mathcal{T}, \quad \{c\} \approx \{p\},$$

so

$$\mathcal{A} \cup \mathcal{Q} \cup \{c\} \approx \mathcal{B} \cup \mathcal{T} \cup \{p\}.$$

Showing $Y \approx U$

$$S = A \cup B \cup C \cup Q, \quad U = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{Q} \cup \{c\}$$

$$X = \mathcal{A} \cup \mathcal{Q} \cup \{c\}, \quad Y = U \setminus X$$

Lemma(3)

If $X_1 \subseteq Y \subseteq X$ with $X \approx X_1$, then $X \approx Y$.

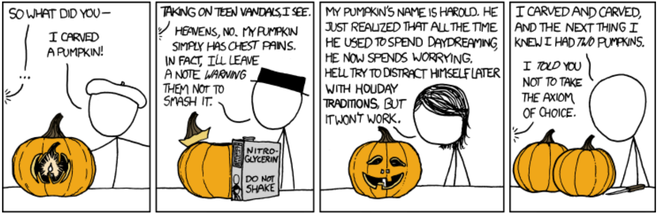
- We have $X \approx \mathcal{B} \cup \mathcal{T} \cup \{p\} \subseteq Y \subset U$, so

$$Y \approx U.$$

Is the Axiom of Choice worth the trouble?

- Before you choose to reject AC, remember that it is equivalent to:
  - the Well-Ordering Theorem
  - the Numeration Theorem
  - the Law of Trichotomy
  - Zorn's Lemma

  all of which are important and useful theorems with wide applications.



xkcd #804 demonstrates how the Banach-Tarski paradox applies to pumpkin carving.